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Equitable Resolvable Coverings

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Abstract: In an earlier article, Willem H. Haemers has determined the minimum number of parallel classes in a resolvable 2-$(qk, k, 1)$ covering for all $k \geq 2$ and $q = 2$ or 3. Here, we complete the case $q = 4$, by construction of the desired coverings using the method of simulated annealing. Secondly, we look at equitable resolvable 2-$(qk, k, 1)$ coverings. These are resolvable coverings which have the additional property that every pair of points is covered at most twice. We show that these coverings satisfy $k < 2q - \sqrt{2q - \frac{9}{4}}$, and we give several examples. In one of these examples, $k > q$. © 2003 Wiley Periodicals, Inc. J Combin Designs 11: 113–123, 2003; Published online in Wiley InterScience (www.interscience.wiley.com). DOI 10.1002/jcd.10024

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1. INTRODUCTION

In this article, we discuss resolvable coverings. A t-(v, k, λ) covering consists of a v-set V (of points) together with a collection of k-subsets of V (called blocks) such that every t-subset of V is a subset of at least λ blocks. A covering is called resolvable if the collection of blocks admits a partitioning into classes (called parallel classes), such that each class consists of $q = v/k$ disjoint blocks [10]. We speak of a minimum resolvable t-(v, k, λ) covering if there exists no resolvable t-(v, k, λ) covering with fewer parallel classes.

Here, we focus on resolvable 2-$(qk, k, 1)$ coverings. Since the case $q = 1$ is trivial, we assume $q \geq 2$ for the remainder of the paper. For small k or q, the problem of finding minimum resolvable coverings has been solved (cf. [5–7]; we warn the reader that in the latter two papers resolvable minimum coverings instead of minimum resolvable coverings were studied). In Section 2.1, we finish the cases with $q = 4$ by

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constructing the desired coverings by the local search method simulated annealing. This method is explained in Section 4.

In Section 2.2, we indicate how to construct minimum resolvable 2-$(\phi^n, \phi^m, 1)$ coverings for any prime power ϕ and $n > m$, by using so-called covers of projective spaces. In these coverings, there are pairs that are covered many times, which is an undesired property in some applications. We, therefore, focus on so-called equitable resolvable coverings in Section 3. These coverings have the property that every pair is covered either once or twice. In Section 3.1, we show that for an equitable resolvable 2-$(qk, k, 1)$ covering to exist, the parameters must satisfy $k < 2q - \sqrt{2q - \frac{9}{4}}$. In Section 3.2, we construct equitable resolvable coverings from affine planes, and in Section 3.3, we focus on equitable resolvable coverings with block size $k \leq 4$. All these coverings have $k \leq q$. We have, however, found one example of an equitable resolvable covering with $k > q$, i.e., a 2-$(20, 5, 1)$ covering. This covering was found by simulated annealing, and as far as we know, it is the only known equitable resolvable covering with $k > q$.

2. MINIMUM RESOLVABLE COVERINGS

Define $r(q, k)$ to be the number of parallel classes in a minimum resolvable 2-$(qk, k, 1)$ covering. It is well known that $r(q, k) \geq q + (q - 1)/(k - 1)$ with equality if and only if there exists a resolvable 2-$(qk, k, 1)$ design. If $k = 2$, a resolvable covering is just a 1-factorization of the complete graph, hence $r(q, 2) = 2q - 1$. If $k = 3$ or 4, $r(q, k)$ is determined for almost all values of q, see [6,7]. An affine plane of order q clearly is a resolvable 2-$(q^2, q, 1)$ covering, so $r(q, q) = q + 1$ if q is a prime power.

2.1. Small Parallel Classes

In [5], the minimum number of parallel classes $r(3, k)$ in a resolvable 2-$(3k, k, 1)$ covering was determined for all k (see below). Here, we complete the determination of the numbers $r(4, k)$ for all k. For this, we first need the following results from [5].

Take a resolvable t-(v, k, λ) covering and replace every point by a set of size m (say), then one obtains a t-(vm, km, λ) covering. Hence:

Lemma 2.1. *If m is a positive integer then $r(q, km) \leq r(q, k)$.*

So $r(q, qm) = q + 1$ if q is a prime power. The next result from [5] characterizes this case.

Proposition 2.1. *The number of parallel classes in a minimum resolvable 2-$(qk, k, 1)$ covering is at least $q + 1$. Equality holds if and only if q divides k and q is the order of an affine plane.*

We also quote the following result from [5].

Proposition 2.2. *Let q be the order of an affine plane and let k be a positive integer such that*

$$\left\lceil \frac{k}{q} \right\rceil \leq \frac{2k}{2q - 1}. \tag{1}$$

Then, there exists a resolvable 2-$(qk, k, 1)$ covering with $q + 2$ parallel classes.

For a given prime power q, Propositions 2.1 and 2.2 give the value of $r(q,k)$ for almost all k. Indeed, for the inequality (1) to hold it suffices that $k \geq 2(q-1)^2$. In particular, for $q=2$, there are no exceptions. If $q=3$, then only for $k=2$, 4, and 7 condition (1) is not satisfied. We know $r(3,2)=5$ and, by Lemma 2.1 and Proposition 2.1, $r(3,4)=5$. A resolvable covering with $q=3$, $k=7$, and $r=5$ was constructed in [5], which proves that $r(3,7)=5$.

Take $q=4$. Then, condition (1) is satisfied for $k \notin \{2,3,5,6,9,10,13,17\}$. We know $r(4,2)=7$. Assaf, Mendelsohn, and Stinson [2] showed that $r(4,3)>6$, and a resolvable 2-$(12,3,1)$ covering with $r=7$ is easy to construct (see for example Proposition 3.1). So $r(4,3)=7$. For $k=5$, 6, 9, 13, and 17 a resolvable covering with $r=6$ is exhibited in the Appendix. These coverings were found by computer using the local search method of simulated annealing (see Section 4). By Lemma 2.1, there also exists a 2-$(40,10,1)$ covering with $r=6$. With Proposition 2.1, this implies that $r(4,k)=6$ for $k \in \{5,6,9,10,13,17\}$. Thus, we have the following result.

Theorem 2.1. *The number $r(4,k)$ of parallel classes in a minimum resolvable 2-$(4k,k,1)$ covering satisfies*

$$r(4,k) = \begin{cases} 7 & \textit{if } k=2 \textit{ or } k=3, \\ 5 & \textit{if } k \equiv 0 \pmod 4, \\ 6 & \textit{otherwise}. \end{cases}$$

2.2. A Construction From Covers of Projective Spaces

Consider the affine space $V = AG(n,\phi)$ (ϕ a prime power) together with a subspace $W = AG(m,\phi)$. Take $x \in V$. Clearly, a translate $W+x$ of W is either equal to W or disjoint from W. So the translates form a partition of V. Take a set $\Omega = \{W_1,\ldots,W_r\}$ of subspaces $AG(m,\phi)$, such that every point of V is in at least one such a subspace, then every pair of points from V is covered by a translate of some W_i. So, Ω and its translates form the blocks of a resolvable 2-$(kq,k,1)$ covering with $k=\phi^m$, $q=\phi^{n-m}$, and r parallel classes. In the corresponding projective space $PV = PG(n-1,\phi)$, the set Ω corresponds to a set $P\Omega$ of subspaces $PW = PG(m-1,\phi)$ that cover all points of PV. Beutelspacher [3] proved that there always exists such a set $P\Omega$ with cardinality

$$r = |P\Omega| = \phi^{\rho+1} \cdot \frac{\phi^{\kappa m}-1}{\phi^m-1} + 1,$$

where κ and ρ are integers such that $n-1 = \kappa m + \rho$ with $0 \leq \rho < m$. Straightforward computations give that $r = q + \lceil (q-1)/(k-1) \rceil$, which implies that the constructed covering is minimal.

The coverings of Beutelspacher have the property that there is a subspace (of dimension $m-2-\rho$) whose $\frac{\phi^{m-1-\rho}-1}{\phi-1}$ points are covered $\phi^{\rho+1}+1$ times, and all other points are covered once. Eisfeld [4] proved that any (minimum) covering of the points of PV with r PW's must be of this form.

3. EQUITABLE RESOLVABLE COVERINGS

In many of the constructions of coverings, there are pairs that are covered many times, like in the construction with covers of projective spaces in Section 2.2. In some applications, this property is undesired. For these applications, we want the pairs to be covered as equally as possible. Coverings having this property are called equitable. More formally, a 2-$(v, k, 1)$ covering is called equitable if every pair is covered once or twice. In this section, we shall first derive some bounds on the parameters of equitable resolvable coverings. We then give a construction from affine planes, and finally, we focus on those equitable resolvable coverings with blocks of size $k \leq 4$.

3.1. Bounds

We shall see that for resolvable equitable coverings the number of parallel classes r is restricted, and consequently that the block size k is upper bounded by a specific function of q, the size of a parallel class.

Suppose, we have an equitable resolvable 2-$(qk, k, 1)$ covering with r parallel classes. Let x_1 and x_2 be the numbers of pairs that are covered once and twice, respectively. By easy counting arguments, we find that $x_1 + x_2 = \binom{qk}{2}$ and $x_1 + 2x_2 = rq\binom{k}{2}$. Hence, we find that $x_1 = 2\binom{qk}{2} - rq\binom{k}{2}$ and $x_2 = rq\binom{k}{2} - \binom{qk}{2}$. Since these numbers must be nonnegative, we find the following constraints:

$$\frac{qk-1}{k-1} \leq r \leq 2 \cdot \frac{qk-1}{k-1},$$

where the extremes are attained when the covering is a resolvable design with $\lambda = 1$ or $\lambda = 2$, respectively (as we have seen earlier, the lower bound holds for all resolvable coverings).

Next, we shall give a lower bound on x_2. This lower bound provides a tighter constraint on r in the case $k > q$.

Lemma 3.1. *For an equitable resolvable* 2-$(qk, k, 1)$ *covering with* r *parallel classes, we have*

$$rq\binom{k}{2} - \binom{kq}{2} \geq q\binom{r}{2}\left(\binom{m}{2}q + m(k - mq)\right),$$

where $m = \lfloor \frac{k}{q} \rfloor$. *Moreover, every block contains at least* $(r-1)\left(\binom{m}{2}q + m(k-mq)\right)$ *pairs that are covered twice.*

Proof. Consider, in such a covering, a block B and a parallel class C not containing B. Let $a_1, a_2, \ldots, a_q$ be the intersection sizes of B with the q blocks of C. It is clear that $a_1 + a_2 + \cdots + a_q = k$, and that the number of pairs in B that are also covered by the blocks in C is equal to $\binom{a_1}{2} + \binom{a_2}{2} + \cdots + \binom{a_q}{2}$. This number is minimal (under the constraint $a_1 + a_2 + \cdots + a_q = k$) when $k - mq$ of the numbers a_i are equal to $m + 1$, and the remaining numbers a_i are equal to m. Thus, it follows that the number of pairs in B that are also covered by the blocks in C is at least $\binom{m}{2}q + m(k - mq)$. Since there are $r - 1$ parallel classes not containing B, and since each pair is covered at most twice, this proves the second statement of the lemma. It also follows that the number of pairs that are covered twice is at least $q\binom{r}{2}\left(\binom{m}{2}q + m(k-mq)\right)$, which proves the first statement. □

For fixed integers q and r, we let

$$s(k) = rq\binom{k}{2} - \binom{kq}{2} - q\binom{r}{2}\left(\binom{m}{2}q + m(k-mq)\right),$$

where $m = m(k) = \lfloor \frac{k}{q} \rfloor$. Thus, if an equitable resolvable 2-$(qk, k, 1)$ covering with r parallel classes exists, then $s(k) \geq 0$.

Lemma 3.2. *Let q, r be integers such that $r \geq q+1$. Then, $s(k)$ is a decreasing function of the integer k, for $k > q$.*

Proof. If k increases by one, then $s(k)$ increases by $\Delta s = rqk - q^2k + \frac{1}{2}q - \frac{1}{2}q^2 + \frac{1}{2}qrm - \frac{1}{2}qr^2m$, (independent of Δm). To show that $\Delta s < 0$, we first decide when Δs is maximal.

If k increases by one, and m does not change, then Δs increases by $\Delta^2 s = rq - q^2 > 0$, so Δs increases. However, if k increases by q, and hence m increases by one, then Δs increases by $\Delta^2 s = rq^2 - q^3 - \frac{1}{2}qr^2 + \frac{1}{2}qr$. This expression is decreasing in r for $r \geq q+1$, hence it follows that $\Delta^2 s \leq -\frac{1}{2}q^3 + \frac{1}{2}q^2 < 0$, since $q \geq 2$. Hence, here Δs is decreasing.

Thus, we may conclude that Δs is maximal for $k = 2q-1$, hence $\Delta s \leq 2rq^2 - \frac{1}{2}rq - 2q^3 + \frac{1}{2}q^2 + \frac{1}{2}q - \frac{1}{2}qr^2$. The latter expression (seen as function of (the integer) r) is maximal for $r = 2q$, hence $\Delta s \leq -\frac{1}{2}q^2 + \frac{1}{2}q < 0$, since $q \geq 2$.

So, if k increases, then $s(k)$ decreases. □

Theorem 3.1. *If an equitable resolvable 2-$(qk, k, 1)$ covering exists, then $k < 2q - \sqrt{2q - \frac{9}{4}}$.*

Proof. For the readability of our proof, we let $\theta = \sqrt{2q - \frac{9}{4}}$. We shall prove that $s(k) < 0$ for all integers $k \geq 2q - \theta$, and all $r \geq q+1$.

First, we shall show that $s(2q-\theta) < 0$ for all r. Since $s(k)$ is a quadratic expression in r, it suffices to show that the corresponding discriminant $D = (-(k-q) - k(k-1))^2 - 4(k-q)(k^2q - k)$ (where we substituted $m = 1$) is negative. Working out D for $k = 2q - \theta$ gives $D = -3q\theta + \frac{7}{2}q - \frac{63}{16}$, which is indeed negative for $q \geq 2$.

There is one obstacle left before we can apply Lemma 3.2, and that is the integrality condition on k. To prove that $s(\lceil 2q - \theta \rceil) < 0$ for all r, we shall show that $s(k)$ is a decreasing function of the real variable k on the interval from $2q - \theta$ to $2q - \theta + 1$.

On the specific interval, the value of m equals one (i.e., it doesn't change). It follows that $s'(k) = \frac{1}{2}q(2k(r-q) - r^2 + 1)$, which is at most $\frac{1}{2}q(-r^2 + 2r(2q - \theta + 1) + 1 - 2q(2q - \theta + 1))$ on the specific interval. The latter expression is quadratic in r, and it follows that $s'(k) < 0$ (on the interval) if the corresponding discriminant is negative. This discriminant equals $q^2(-2(q+1)\theta + 4q - \frac{1}{4})$ which is indeed negative for $q \geq 2$. Thus $s(k)$ is decreasing on the specific interval, and hence $s(\lceil 2q - \theta \rceil) < 0$ for all r.

Now Lemma 3.2 applies, which finishes the proof. □

The question remains how tight the inequality of Theorem 3.1 is. For resolvable 2-(qk, k, λ) designs with $\lambda = 1$ or 2, we know that $k \le q$ (for $\lambda = 1$, this is trivial; for $\lambda = 2$, this follows for example from Theorem 3.1 and the integrality condition on $r = 2(qk-1)/(k-1)$). For equitable resolvable coverings in general, this is, however, not true. By simulated annealing, we found a resolvable equitable 2-$(20, 5, 1)$ covering with 7 parallel classes (by Lemma 3.1 this is a minimum equitable resolvable covering). This covering is contained in Table A.6 of the Appendix. As far as we know, it is the only known equitable resolvable covering with $k > q$.

3.2. Constructions From Affine Planes

In this section, we give a construction of equitable resolvable coverings from affine planes. For some values of k and q, these coverings have the minimum number $(q+2)$ of parallel classes. The construction is a generalization of the 2-$(20, 4, 1)$ covering with 7 parallel classes constructed by Lamken, Mills, and Rees [7].

Proposition 3.1. *Let q be the order of an affine plane, and let $k = q - t$ be such that $\frac{2}{3}q \le k < q$. Then, there exists an equitable resolvable* 2-$(qk, k, 1)$ *covering with $q + 3$ parallel classes. If, moreover, $q \equiv t(\text{mod } 2t)$, then there exists an equitable resolvable* 2-$(qk, k, 1)$ *covering with $q + 2$ parallel classes.*

Proof. Take an affine plane of order q. Fix a parallel class C, and k of its lines. Label these lines by $l_i, i = 1, \ldots, k$. Fix another parallel class C', and label its lines by $m_j, j = 1, \ldots, q$. As point set of the covering, we take all points on the lines $l_i, i = 1, \ldots, k$. We label these points by the pairs $(i,j), i = 1, \ldots, k; j = 1, \ldots, q$, where (i,j) is the point of intersection of l_i and m_j.

The first q parallel classes of the covering are inherited from the affine plane; they are the parallel classes different from C (induced on the new point set). Thus, these parallel classes consist of q blocks of size k; and all pairs of points $\{(i,j), (i',j')\}$ with $i \ne i'$ are covered exactly once by the blocks in these parallel classes.

To cover the remaining pairs, we construct three additional parallel classes. The first parallel class consists of the blocks

$$\{(i,j) : j = 1, \ldots, k\}, i = 1, \ldots, k \text{ and } \{(i,j) : i = 1, \ldots, k\}, j = k+1, \ldots, q,$$

while the second consists of the blocks

$$\{(i,j) : j = t+1, \ldots, q\}, i = 1, \ldots, k \text{ and } \{(i,j) : i = 1, \ldots, k\}, j = 1, \ldots, t.$$

The final parallel class consists of the blocks

$$\{(i,j) : j = 1, \ldots, t, k+1, \ldots, q\} \cup \{(i+h, t+h) : h = 1, \ldots, k-2t\}, i = 1, \ldots, k$$

(where $i + h$ should be considered modulo k) and $\{(i,j) : i = 1, \ldots, k\}, j = k - t + 1, \ldots, k$.

It is easy to check now that this construction gives an equitable 2-$(qk, k, 1)$ covering (with $q + 3$ parallel classes).

If, moreover, $q \equiv t(\text{mod } 2t)$, then we can do better. In that case $k = q - t \equiv 0$ (mod $2t$), say $k = 2ts$. Now the final parallel class in the above construction is replaced by the parallel class consisting of the blocks

$$\{(i,j) : i = hs+1, \ldots, hs+s; j = 1, \ldots, t, k+1, \ldots, q\}, h = 0, \ldots, 2t-1 \text{ and}$$
$$\{(i,j) : i = 1, \ldots, k\}, j = t+1, \ldots, k.$$

But, now all lines m_j (induced on the new point set) occur twice, so we can (and must) remove the parallel class C', to obtain an equitable resolvable covering with $q+2$ parallel classes. □

We remark that the additional restriction $q \equiv t(\text{mod } 2t)$ is for example satisfied when q is odd and $k = q-1$.

3.3 Small Blocks

For small block size k, i.e., for $k \leq 4$, and for almost all q, resolvable 2-$(qk, k, 1)$ coverings with minimal r have been constructed. Many of those are equitable, as we shall see.

Consider a resolvable 2-$(qk, k, 1)$ covering with r parallel classes, and fix a point p. Let y_i be the number of pairs on p that are covered i times. Then, $\sum_i y_i = v-1$ and $\sum_i iy_i = r(k-1)$, hence $\sum_i (i-1)y_i = r(k-1) - v + 1$. Now, suppose that $v = r(k-1)$, then it follows that $y_i = 0$ for $i > 2$, and $y_2 = 1$. This implies that the covering is equitable, and that every point is in exactly one pair which is covered twice.

For $k = 3$, there exist 2-$(3q, 3, 1)$ coverings with $r = \frac{3}{2}q$ for all even $q \neq 2, 4$ [6]. These coverings are equitable by the above arguments. For $q = 2$ there can be no equitable covering by the results in Section 3.1. For $q = 4$, a covering with 6 parallel classes does not exist (cf. [2]); however, the construction of Proposition 3.1 gives an equitable resolvable covering with 7 parallel classes. We remark that also for all odd q there exist 2-$(3q, 3, 1)$ coverings with $r = \frac{3q-1}{2}$. These coverings are designs (Kirkman triple systems), and hence they are equitable.

For $k = 4$, there exist 2-$(4q, 4, 1)$ coverings with $r = \lceil \frac{4q-1}{3} \rceil$ for all but finitely many q [7]. For $q \equiv 1 \pmod 3$, these coverings are designs, and hence they are equitable. For $q \equiv 0 \pmod 3$, the coverings are equitable by the above arguments. For $q \equiv 2 \pmod 3$, the coverings may or may not be equitable. From the above arguments, it follows that in these coverings each point is in two pairs that are covered twice, or in one pair that is covered three times. Typically, these coverings as constructed by Lamken, Mills, and Rees [7] are not equitable (the coverings with $q \in \{5n, 11n, 17n, 23n : n \equiv 1 \pmod 3\} \cup \{47, 59, 71, 83, 107, 131, 179\}$ are equitable). For $q = 2$, no equitable resolvable covering exists (but a resolvable covering with $r = 3$ exists). For $q = 5$, the covering from [7] (and Proposition 3.1) is equitable. The smallest open case is $q = 8$: the covering (with $r = 11$) from [7] is not equitable, just like the covering obtained from the cover of $PG(4, 2)$ with (projective) lines (see Section 2.2).

In general, Beutelspacher's [3] cover of $PG(n-1, 2)$, n odd, by lines gives a resolvable 2-$(2^n, 4, 1)$ covering with $\frac{2^n+1}{3}$ parallel classes that is not equitable. It is, however, easy to adjust this cover of $PG(n-1, 2)$ such that every point is covered once or twice. To do this, take one of the lines that covers the unique point that is covered three times, and remove this line l. Now add two lines that do not intersect, each line covering one of the two points that were uncovered by the removal of the line l. By Eisfeld's [4] characterization, this is a minimum "equitable" cover of $PG(n-1, 2)$ with lines. The corresponding equitable resolvable 2-$(2^n, 4, 1)$ covering with $\frac{2^n+1}{3} + 1$ parallel classes may, however, not be minimal (in the sense that we do not know if there exists an equitable resolvable 2-$(2^n, 4, 1)$ covering with $\frac{2^n+1}{3}$ parallel classes).

4. SIMULATED ANNEALING

Simulated annealing is a local search method [1], which has proved to be fruitful for the construction of coverings [8]. Here, we apply the technique in the following way. Suppose we want to find a resolvable 2-$(qk, k, 1)$ covering with r parallel classes for fixed k, q, and r. We search in the set of resolvable 1-(qk, k, r) designs (i.e., r parallel classes consisting of q blocks of size k). We define the costs $f(\mathcal{D})$ of such a design $\mathcal{D}$ to be the number of pairs of points from $\mathcal{D}$ not covered by the blocks of $\mathcal{D}$. Clearly, if $f(D) = 0$, then $\mathcal{D}$ is a resolvable covering. A neighbor of $\mathcal{D}$ is obtained by interchanging two points from different blocks within a parallel class. A neighbor $\mathcal{D}'$ of $\mathcal{D}$ is accepted as a next item in the search with probability

$$p_{c_i} = \begin{cases} 1 & \text{if } f(\mathcal{D}') \leq f(\mathcal{D}), \\ e^{(f(\mathcal{D}) - f(\mathcal{D}'))/c_i} & \text{if } f(\mathcal{D}') > f(\mathcal{D}). \end{cases}$$

The parameter c_i is the control parameter which depends on the number of steps i in the search. It converges to 0 as $i \to \infty$. We have successfully applied the search with $q = 4$, $r = 6$, $k \in \{5, 6, 9, 13, 17\}$, and control parameter $c_i = \frac{1}{2}(0.99)^{\lfloor 10^{-4} i \rfloor}$. The results are exhibited in Tables A.1.–A.5 of the Appendix. For $k = 5$ and $k = 6$, the solution was found by the computer in less than a minute. For $k \in \{9, 13, 17\}$, a first attempt by computer was not successful. Only after finding a good resolvable 1-design to start with, the solutions were found; also here within a minute. For $k = 9$, this good starting 1-design was found by combining the solution for $k = 5$ and the optimal solution for $k = 4$, the affine plane of order 4 (a resolvable 2-$(16, 4, 1)$ covering with $r = 5$). First, each of the 5 parallel classes of the affine plane was united with one of the (6) parallel classes of the resolvable 2-$(20, 5, 1)$ covering. Within the 5 "new" parallel classes, each block of size 4 was united with a block of size 5 to obtain blocks of size 9. This was done by simulated annealing to get many pairs covered. Secondly, the 16 points of the affine plane were distributed over the blocks of the remaining parallel class of the 2-$(20, 5, 1)$ covering, so that also these blocks got size 9. Also this distribution was done by simulated annealing to get many pairs covered. The resulting resolvable 1-$(36, 9, 6)$ design was then successfully used as a starting point for the initial method. Similarly, a combination of the solution for $k = 9$ and the affine plane gave rise to a good starting point for $k = 13$. Perhaps surprisingly, the starting point for $k = 17$, obtained by a combination of the solution for $k = 13$ and the affine plane, was a resolvable covering already. For more details, see [9].

By the same method, we searched for equitable resolvable coverings. The only difference with the above is the cost function. For a resolvable 1-design $\mathcal{D}$, let x_1 denote the number of pairs that are covered once, and let x_2 be the number of pairs that are covered twice. We saw that for an equitable resolvable covering, the numbers x_1 and x_2 are determined by k, q, and r. Denote these numbers by $\hat{x}_1$ and $\hat{x}_2$. The costs of a resolvable 1-design are now given by $|x_1 - \hat{x}_1| + |x_2 - \hat{x}_2|$. We applied the method for $k = 5$, $q = 4$ and $r = 7$ with the same control parameter as above. After several hours of searching, the computer found the solution given in Table A.6.

APPENDIX

Table A.1. A Minimum Resolvable 2-(20,5,1) Covering

18	20	3	2	17	16	18	1	14	3	10	6	18	13	12
11	16	6	15	12	11	5	20	10	4	19	17	3	8	11
7	9	14	8	10	17	12	19	9	15	14	20	7	1	15
5	4	13	1	19	6	7	2	8	13	2	4	5	9	16
5	4	3	12	7	7	18	19	11	2	13	17	7	16	20
6	16	19	20	8	5	14	17	4	6	14	10	19	12	2
10	17	1	2	15	8	20	1	9	12	5	18	4	15	8
9	11	13	18	14	13	16	3	15	10	6	1	3	11	9

Table A.2. A Minimum Resolvable 2-(24,6,1) Covering

3	14	8	24	16	7	11	17	7	23	20	16
15	6	10	21	17	11	12	6	14	10	18	3
5	23	22	12	4	1	5	19	2	9	22	13
13	18	2	20	9	19	4	24	21	1	8	15
9	10	4	6	13	24	21	18	8	4	22	20
20	23	12	15	5	8	14	23	9	15	3	13
7	18	11	17	16	22	10	5	6	7	16	1
1	3	21	19	14	2	11	19	12	17	24	2
2	19	10	8	23	6	4	2	7	15	16	19
13	16	9	7	12	21	14	3	22	6	10	20
22	20	15	24	1	18	8	17	13	1	11	9
5	17	3	14	11	4	24	21	18	5	23	12

Table A.3. A Minimum Resolvable 2-(36,9,1) Covering

12	6	11	8	27	32	3	31	28	10	12	1	7	17	33	19	24	27
1	2	15	9	23	25	30	24	26	14	4	2	11	18	20	21	25	32
13	16	10	4	21	20	29	17	35	16	5	9	6	3	36	23	29	31
5	7	14	22	34	36	19	18	33	8	13	30	15	26	22	34	35	28
19	13	6	7	3	2	35	25	31	33	2	12	16	25	27	29	34	30
11	23	10	22	32	34	17	30	9	13	5	11	1	22	32	35	36	24
12	5	4	15	27	21	36	26	20	10	6	14	15	26	3	17	18	31
16	18	1	33	28	8	14	24	29	8	7	9	4	21	20	19	28	23
11	7	16	15	29	32	19	24	26	26	15	13	9	35	33	11	23	32
8	2	5	10	34	36	17	25	28	17	29	27	2	22	12	16	8	25
14	9	13	12	23	1	35	27	18	5	10	18	14	30	7	19	28	36
31	4	6	3	21	30	33	20	22	1	4	3	31	6	34	20	21	24

Table A.4. A Minimum Resolvable 2-(52,13,1) Covering

12	6	11	8	21	23	30	38	50	52	35	34	49
1	2	15	9	17	18	31	25	39	41	46	40	42
13	16	10	4	29	32	26	20	37	36	45	33	51
5	7	14	3	28	22	27	24	43	48	19	47	44
10	12	1	7	24	29	46	31	42	38	50	51	44
14	4	2	11	30	20	18	27	34	36	37	41	48
16	5	9	6	32	21	25	22	19	52	39	45	47
8	13	3	15	26	28	17	23	33	49	35	40	43
8	14	16	1	27	23	32	31	45	48	35	40	42
6	2	13	7	24	18	21	26	50	52	33	41	44
12	5	4	15	47	20	22	19	37	46	49	36	38
11	10	3	9	30	25	29	28	39	17	51	43	34
3	2	12	16	49	18	28	32	41	43	45	50	46
13	5	11	1	26	22	30	31	42	19	33	34	47
10	6	14	15	29	21	27	17	38	48	51	52	40
8	7	9	4	24	23	25	20	37	36	35	44	39
11	7	16	15	32	34	17	49	44	24	30	40	45
8	2	5	10	35	29	22	23	19	18	51	41	47
14	9	13	12	27	39	26	38	48	50	33	46	25
1	4	6	3	28	21	20	31	43	37	52	42	36
8	7	11	6	21	26	34	30	46	23	35	44	52
16	13	2	3	33	45	43	18	38	28	32	24	41
15	4	12	5	17	20	19	47	22	50	36	37	40
10	14	9	1	42	31	29	25	51	49	27	39	48

Table A.5. A Minimum Resolvable 2-(68,17,1) Covering

12	6	11	8	28	22	27	24	37	39	46	54	66	68	51	50	65
1	2	15	9	17	18	31	25	33	34	47	41	55	57	62	56	58
13	16	10	4	29	32	26	20	45	48	42	36	53	52	61	49	67
5	7	14	3	21	23	30	19	44	38	43	40	59	64	35	63	60
10	12	1	7	26	28	17	23	40	45	62	47	58	54	66	67	60
14	4	2	11	30	20	18	27	46	36	34	43	50	52	53	57	64
16	5	9	6	32	21	25	22	48	37	41	38	35	68	55	61	63
8	13	3	15	24	29	19	31	42	44	33	39	49	65	51	56	59
8	14	16	1	24	30	32	17	43	39	48	47	61	64	51	56	58
6	2	13	7	22	18	29	23	40	34	37	42	66	68	49	57	60
12	5	4	15	28	21	20	31	63	36	38	35	53	62	65	52	54
11	10	3	9	27	26	19	25	46	41	45	44	55	33	67	59	50
3	2	12	16	19	18	28	32	65	34	44	48	57	59	61	66	62
13	5	11	1	29	21	27	17	42	38	46	47	58	35	49	50	63
10	6	14	15	26	22	30	31	45	37	43	33	54	64	67	68	56
8	7	9	4	24	23	25	20	40	39	41	36	53	52	51	60	55
11	7	16	15	27	23	32	31	48	50	33	65	60	40	46	56	61
8	2	5	10	24	18	21	26	51	45	38	39	35	34	67	57	63

Table A.5. (*Continued*)

14	9	13	12	30	25	29	28	43	55	42	54	64	66	49	62	41
1	4	6	3	17	20	22	19	44	37	36	47	59	53	68	58	52
8	6	7	11	24	23	27	22	37	42	50	46	62	39	51	60	68
16	13	2	3	32	29	18	19	49	61	59	34	54	44	48	40	57
5	15	4	12	31	20	28	21	33	36	35	63	38	66	52	53	56
1	14	9	10	26	30	25	17	58	47	45	41	67	65	43	55	64

Table A.6. An Equitable Resolvable 2-(20,5,1) Covering with $r = 7$

8	4	6	16	3	5	20	18	7	16	10	17	15	3	18
13	18	1	14	19	6	10	8	13	11	12	1	6	13	7
9	7	17	15	11	12	14	17	4	9	20	2	14	9	8
10	5	2	20	12	19	2	3	1	15	5	16	11	4	19
8	16	19	12	15	9	3	12	16	13	17	11	2	16	13
17	5	2	6	14	18	5	1	8	17	4	15	1	20	12
18	20	7	13	4	2	10	19	4	7	3	7	8	5	14
11	1	3	9	10	20	15	14	11	6	6	19	10	9	18
1	16	14	7	10										
18	12	8	11	2										
3	20	17	6	19										
4	5	15	13	9										

REFERENCES

[1] E. H. L. Aarts, J. H. M. Korst, and P. J. M. van Laarhoven, Simulated annealing, In: Local Search in Combinatorial Optimization (E. Aarts and J. K. Lenstra, eds.), John Wiley & Sons, 1997, pp. 91–120.

[2] A. Assaf, E. Mendelsohn, and D. R. Stinson, On resolvable coverings of pairs by triples, Utilitas Mathematica 32 (1987), 67–74.

[3] A. Beutelspacher, On *t*-covers in finite projective spaces, J Geometry 12 (1979), 10–16.

[4] J. Eisfeld, On smallest covers of finite projective spaces, Archiv Math 68 (1997), 77–80.

[5] W. H. Haemers, Minimum resolvable coverings with small parallel classes, Discrete Math 197/198 (1999), 393-396.

[6] E. R. Lamken and W. H. Mills, Resolvable coverings, Congressus Numerantium 96 (1993), 21–26.

[7] E. R. Lamken, W. H. Mills, and R. S. Rees, Resolvable minimum coverings with quadruples, J Combinatorial Designs 6 (1998), 431–450.

[8] K. J. Nurmela and P. R. J. Östergård, Upper bounds for covering designs by simulated annealing, Congressus Numerantium 96 (1993), 93–111.

[9] M. B. M. Peek, Minimum resolvable coverings, Master's thesis (in Dutch), Tilburg University, 2000.

[10] D. R. Stinson, Coverings, Chapter IV-8 in The CRC Handbook of Combinatorial Designs (C. J. Colbourn and J. H. Dinitz, eds.), CRC Press, 1996, pp. 260–265.